## Lecture 13 Highlights

Phys 402
Up to this point, we have only considered static solutions to the Schrödinger equation. It is now time to consider what happens to a quantum system when it is given a time-dependent perturbation. The philosophy of this calculation is as follows. Consider a quantum system governed by a time-independent 'baseline' or unperturbed Hamiltonian $H^{0}$ that has solutions to the time-dependent Schrödinger equation $H^{0} \Psi_{n}^{0}(\vec{r}, t)=i \hbar \frac{d}{d t} \Psi_{n}^{0}(\vec{r}, t)$ of the form $\Psi_{n}^{0}(\vec{r}, t)=\phi_{n}^{0}(\vec{r}) e^{-i E_{n}^{0} t / \hbar}$, where $E_{n}^{0}$ is the unperturbed eigen-energy. Suppose that this system is prepared in a particular eigenstate, say the $n{ }^{\text {th }}$ state (where ' $n$ ' is a list of quantum numbers in general). Next consider turning on a "small" time-dependent perturbing potential such that the new Hamiltonian is given by $\mathcal{H}^{0}+\lambda \mathcal{H}^{1}(\vec{r}, t)$, where $\lambda \ll 1$ and the perturbation is in general a function of both position and time. Let this perturbation act for some time ' $t$ ', and then have it stop. Now the system is governed once again by the unperturbed time-independent Hamiltonian $H^{0}$. The question is this: what is the probability that the quantum system is now in some other state " $j$ "? This is equivalent to asking for the probability that the system has made a quantum jump from state ' $n$ ' to state ' $j$ '.

To address this question we employ a time-dependent version of perturbation theory. While the perturbation is on, the wavefunction becomes $\Psi(\vec{r}, t)$ and satisfies the new time-dependent Schrödinger equation:

$$
\left[H^{0}+\lambda H^{\prime}(\vec{r}, t)\right] \Psi(\vec{r}, t)=i \hbar \frac{d}{d t} \Psi(\vec{r}, t)
$$

We employ the trick of expanding the new wavefunction around the unperturbed solution plus a series of ever smaller corrections, $\Psi_{n}=\Psi_{n}^{0}+\lambda \Psi_{n}^{1}+\lambda^{2} \Psi_{n}^{2}+\ldots$, and substitute this into the time-dependent Schrödinger equation. Collecting like-powers of $\lambda$ yields
$\lambda^{0}: H^{0} \Psi_{n}^{0}=i \hbar \frac{d}{d t} \Psi_{n}^{0}$, which is the original unperturbed problem, $\lambda^{1}: H^{0} \Psi_{n}^{1}+H^{\prime} \Psi_{n}^{0}=i \hbar \frac{d}{d t} \Psi_{n}^{1}$. We use the completeness postulate of quantum mechanics to express the first order correction to the wavefunction as an infinite sum over all the unperturbed eigenfunctions: $\Psi_{n}^{1}=\sum_{\ell} a_{n l}(t) \Psi_{\ell}^{0}(\vec{r}, t)$ with unknown time-dependent coefficients $a_{n l}(t)$. Substituting this into the $\lambda^{1}$ equation and projecting out the (arbitrarilychosen) $j^{\text {th }}$ eigenstate yields the amplitude transition rate from state ' $n$ ' to state ' j ':

$$
\begin{equation*}
\dot{a}_{n j}=\frac{-i}{\hbar} e^{i\left(E_{j}^{0}-E_{n}^{0}\right) t / \hbar} \int \phi_{j}^{0^{*}}(\vec{x}) \mathrm{H}^{\prime}(\vec{x}, t) \phi_{n}^{0}(\vec{x}) d^{3} x \tag{1}
\end{equation*}
$$

Hence if we know the perturbing Hamiltonian, this matrix element can be computed and the result integrated over time to find the transition amplitude from state ' $n$ ' to state ' j ', $a_{n j}(t)$. The probability of the transition is proportional to $\left|a_{n j}(t)\right|^{2}$.

We then considered two-level systems, as discussed by Griffiths in the first few pages of Chapter 11. This is a manageable system that allows us to see how time-dependent perturbation works in a relatively simple, but still very useful, system.

A 2-level system with states ' $a$ ' and ' $b$ ' subject to a time-dependent perturbation will have a wavefunction of the form:

$$
\Psi(t)=c_{a}(t) \psi_{a} e^{-i E_{a} t / \hbar}+c_{b}(t) \psi_{b} e^{-i E_{b} t / \hbar}
$$

Assuming that the system started in state "a" at time $t=0$, just before the time-dependent perturbation began, gives the initial conditions:

$$
c_{a}(0)=1, \quad c_{b}(0)=0 .
$$

Demanding that $\Psi(t)$ satisfies the time-dependent Schrödinger equation, we can solve for the rate at which amplitude builds up in state ' $b$ ':

$$
\dot{c}_{b}(t)=\frac{-i}{\hbar} e^{i\left(E_{b}-E_{a}\right) t / \hbar} \int \psi_{b}^{*}(\vec{x}) H^{\prime}(\vec{x}, t) \psi_{a}(\vec{x}) d^{3} x
$$

This result is a special case of Eq. (1) above.


The quantum state of a two-level system (TLS) is specified nicely by a point on the Bloch sphere. A two-level system has a two-dimensional Hilbert space. This can be represented as a point on a sphere of unit radius in 3 -space. One can prepare the system in a superposition of the two states as, $|\Psi\rangle=c_{a}(0)|0\rangle+c_{b}(0)|1\rangle$, where $|0\rangle$ represents the ground state and $|1\rangle$ is the excited state, and the $c$ 's are complex numbers in general. By mapping this on to the unit sphere, we can represent any state of this TLS in terms of the "latitude" and "longitude" angles as $|\Psi\rangle=\cos \frac{\theta}{2}|0\rangle+e^{i \phi} \sin \frac{\theta}{2}|1\rangle$. Note that this wavefunction is normalized. The two pure states $|0\rangle$ and $|1\rangle$ are given by $\theta=0$ and $\theta=$ $\pi$, respectively. Any possible superposition can be represented with general values for the two angles (see the class web site for examples). Starting from the north pole of the Bloch sphere (i.e. state $|0\rangle$ ) one can perform two operations to get to any other possible state: i) Rotate about the x -axis (or y-axis) by an angle $\theta$, and ii) Rotate about the z -axis by angle $\phi$. Moving from the north pole to the south pole is called a NOT operation, and can be accomplished with a "Rabi oscillation" process, which we now investigate.

